

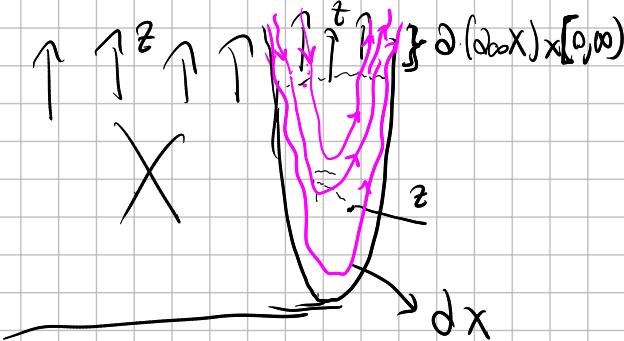
## LIOUVILLE SECTORS, PART 2

Recall def:  $X$  Liouville mfd w. bdry (in part, ask that Liouville v.f.  $\vec{z}$  is tangent to  $\partial X$  on  $N\text{bd}(\infty)$ ) is Liouville sector if

$\exists I: \partial X \rightarrow \mathbb{R}$  with  $\partial I|_{\text{char fd}} > 0$  ( $\Leftrightarrow X_\varepsilon$  outwards pointing)

and  $\partial^I(z) = \alpha I$  for some (an)  $\alpha > 0$  near  $\infty$

Rank It follows from def that  $I$  assumes both pos and neg values



Last week: saw def, examples, construction of product, some links with sutured Liouville domains ( $\hookrightarrow$  Liouville pairs of Etnesberg)

- Plan Today:
  - product decomposition  $F \times G \text{ for } \alpha > 0$  near  $\partial X$
  - convex completion: Liouville sector into L-domain
  - comparison w/ Liouv. parts and strips
  - control J-hol curves
  - control Reeb dynamics

## • Product decomposition near $\partial X$ :

Consider  $F$ ,  $w_0 = dx \wedge dy$ ,  $f = x + iy$ ,  $t_0 = \frac{1}{2}x dx + \frac{1}{2}y dy$

Let  $(X, \lambda)$  Liouville sector

Prop Given  $I: Nbd \rightarrow \mathbb{R}$   $\frac{1}{2}$ -defining function,  
there is a canonical identification  
 $Nbd \partial X \cong Nbd \circ (F \times \mathbb{C}_{Re \geq 0}, \lambda_F + \lambda_C^{i z_0 w_0} + df)$

where: (a)  $I = y$  imaginary coordinate on  $\mathbb{C}_{Re \geq 0}$

(b)  $(F, \lambda_F)$  Liouville manifold

(c)  $f: F \times \mathbb{C}_{Re \geq 0} \rightarrow \mathbb{R}$  satisfies:

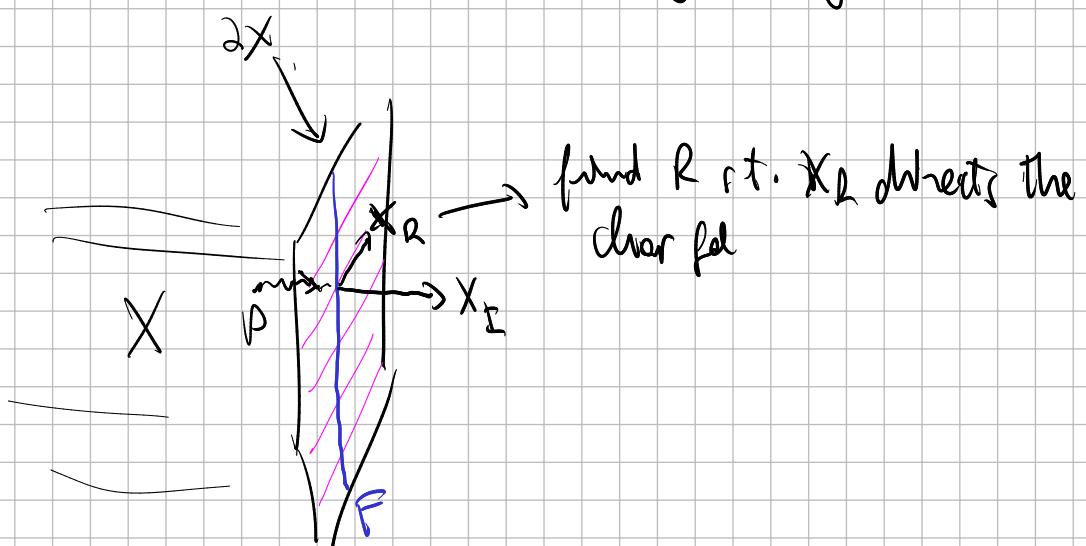
(1)  $f$  has support in  $F_0 \times \mathbb{C}_{Re \geq 0}$

with  $F_0 \subset F$  Liouville domain

(2)  $f$  coincides with some  $f_{\pm \infty}: F \rightarrow \mathbb{R}$

for  $|I|$  big enough

Idea :



Pf Define  $R: \text{Nbd}(\partial X) \rightarrow \mathbb{R}_{\geq 0}$  as  $R|_{\partial X} = 0$

and  $R(p) = " \text{time it takes to reach } \partial X \text{ from } p \text{ flowing } X_I."$

In part  $\underbrace{w(X_2, X_I)}_{z=1} = -dR(X_2) = 1$

$$\Rightarrow dR(z) = \frac{1}{z} R \text{ near } \infty$$

Consider  $\tilde{r}: R + i\mathbb{F}: \text{Nbd}(\partial X) \rightarrow \mathbb{C}_{Re \geq 0}$

This is a sympl fibration with a flat sympl.

Connection  $\text{Vert}^{\perp W} = \langle X_2, X_I \rangle_{\mathbb{R}}$  ( $\text{Vert} = \text{ker } d\pi$ )

By parallel transport, get  $\text{Nbd } \partial X \xrightarrow{\text{sympl}} \text{Nbd } \partial(F \times \mathbb{C}_{Re \geq 0})$

Need to compare Liouville forms:

take  $\lambda_F := \lambda|_{F \times \{0\}}$ , then  $\lambda = \lambda_F + \lambda_C + df$

for some  $f: \text{Nbd}(\partial X) \rightarrow \mathbb{R}$

Notice:

- $d\pi(z) \propto z_C \text{ near } \infty \Rightarrow X_f \text{ tangent to } \text{Vert}_{\text{near } \infty}$

In part,  $f$  satisfies (2)

- $z$  is tangent to  $F_x \text{ by } \{I=0\}$  near  $\infty$

so there  $\lambda = \lambda_F \Rightarrow f$  satisfies (1)



(generalization of) This prep gives:

Cor: Any L-sector may be deformed to make  $\partial X$  exact (we  $f + \omega = f - \omega$  in above prep)

## Convex completion

$X$  (convex sector), identify  $Nbd(\partial X) \geq Nbd \circ (F_X \cap \mathbb{R}_{\operatorname{Re} \leq 0})$

Def Convex completion  $\bar{X} := X \cup \underset{\text{bdry}}{\bigcup_{\operatorname{Re} \leq 0}} F_X \circ \mathbb{C}$

$$\bar{W} := W \cup W_F + W_C$$

Lemma There is a  $\bar{W}$ -Loomis v.f.  $\mathcal{Z}_{\bar{X}}$  making  $(\bar{X}, \bar{W})$  L.wf.

Pf Extend  $f$  on  $Nbd(\partial X)$  or  $F_X \cap \mathbb{R}_{\operatorname{Re} \leq 0}$

so that still satisfies (1), (2)

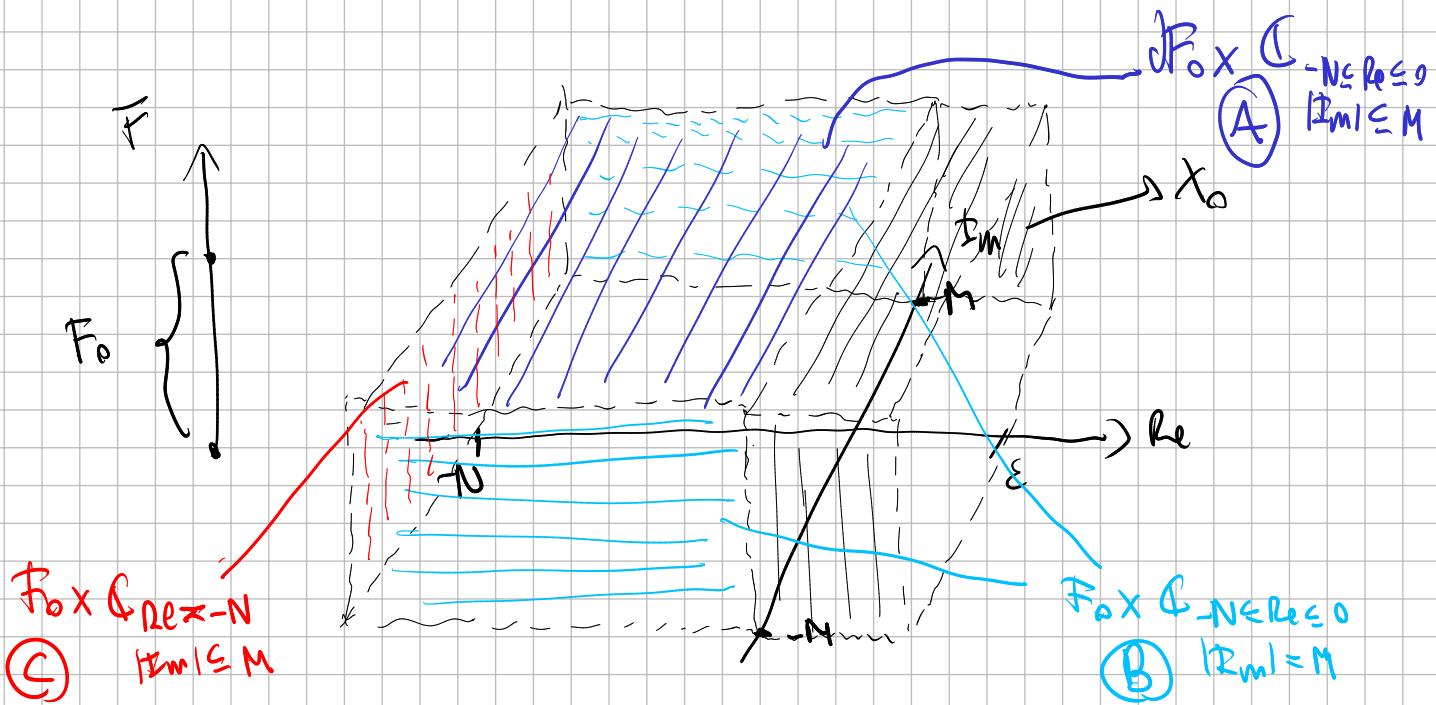
Look at L.wf.  $\mathcal{Z}_{\bar{X}} = \mathcal{Z}_F + \mathcal{Z}_C - X_F$  near

bdry of  $\bar{X}_0 := X_0 \cup \underset{-N \leq \operatorname{Re} \leq 0}{\bigcup_{|\Im m| \leq M}} F_0 \times \mathbb{C} \subset \bar{X}$

for  $M, N \gg 0$ .

$X = X_0 \cup \partial_{\operatorname{Re}} X \times [0, \infty)$   
with  $X_0 = F_0 \times \mathbb{C}$   $\begin{cases} \operatorname{Re} \leq 0 \\ |\Im m| \leq M \end{cases}$

Claim:  $\mathcal{Z}_{\bar{X}}$   $\rightarrow$  bdry of  $\bar{X}_0$  near  $\partial X$



Over A:  $f = 0$  so  $\bar{t}_X = \bar{t}_F + \bar{t}_G$  antwärts pointing.

Over B:  $\bar{t}_X = \bar{t}_F + \bar{t}_G - X$   $\underbrace{f \geq 0}_{\text{antwärts pointing}}$   
tangent to F factor

Over C:  $\bar{t}_X = \bar{t}_F + \bar{t}_G - X_f$  antwärts pointing

iff  $-\frac{\partial f}{\partial x} + N \geq 0$ , hold wif  $N \gg 0$   
(for well chosen  $f$ ).  $\square$



• Comparison with Liouville pairs and stopped L. domains

(a) Lemma | Every Liouv. sector arises from a s.t. L. dom.

Recall: - Sustained Liouv. dom = Liouville pairs  
 $= X_0 \text{ L. dom.} + F_0 \subset \partial X_0 \text{ L. dom}$

- Sustained L. dom.  $\iff (\bar{X}_0, G)$  s.t.

$\bar{X}_0$  Liouv. dom,  
 $\lambda$  Liouville form,  
 $G \subset \partial \bar{X}_0$  cod. 0,  
with  $\exists: G \rightarrow \mathbb{R}$  st.  
 $dI(R_X) > 0$  and  
dt r.f.  $V_I$  f.a.t  $\partial G$

(Idea of pf: In p.w. pict., take  $G$  w.r.t  $\partial \bar{X}_0$ )

(b) Recall stop on Liouville mfld  $\bar{X}$  w.s  $\bar{G}: F\bar{X}(\mathcal{O}_{\bar{X}}) \rightarrow \bar{X}$

proper, and 0 embedding,  $F$  Liouville mfld

with  $\bar{G}^* \lambda = \lambda_F + \lambda_0 + dg$ ,  $g$  compactly supported

If  $(\bar{X}, \bar{G})$  stopped L. mfld,  $\bar{X} \setminus \bar{G}(F\bar{X}(\mathcal{O}_{\bar{X}}))$  is a L. set

(conversely, convex completion  $\bar{X}$  of an exact L. sector

has stop  $\bar{G}: F\bar{X}(\mathcal{O}_{\bar{X}}) \rightarrow \bar{X}$

(need exact to ensure that  $g$  has cpt support)

by taking  $\lambda'_F = \lambda_F + df_{\bar{X} \setminus \bar{G}} = f_{\bar{X} \setminus \bar{G}}$

## Controlling J-hol curves

Two concerns for compactness: (a) escaping at  $\infty$   
 (b) crossing  $\partial X$

(a) We use: (i) monotonicity inequality  
 (ii) pseudoconvexity

(i) Any  $(X, \omega)$  L. sit,  $J$  almgpl cyl compact;

$\Rightarrow$  geometrically bounded

$\hookrightarrow$  (pt: same as for  
 L. mfd's)

$\Rightarrow$  we have monotonicity inequality:

$u: \Sigma \rightarrow X$  Jhol through  $p$

$$\text{Hes } \int_{\partial B_\varepsilon(p)} u^* \omega \geq \text{Const. } \varepsilon^2$$

$\Rightarrow$  Jhol curves w. finite energy  
 cannot escape at  $\infty$

(ii) Lemma Let  $u: \Sigma \rightarrow X = X_0 \cup M \times [0, \infty)$

$J$  hol on  $X$  L. sit,  $J$  cyl, s.t.  
 $u(\partial \Sigma) \subset \overbrace{M \times [0, \infty)}^L$  and  $|u^* \lambda|_{\partial \Sigma} \leq 0$   
 (e.g. Lagr boundary cond on  $L = \mathbb{R} \times [0, \infty)$ )  
 Then  $u$  no loc. const on  $\bar{u}^{-1}(M \times [0, \infty))$ .

(b) preventing crossing of  $\partial X$

Recall  $Nbd(\partial X) \cong Nbd \circ (F_X \cap_{Re \geq 0}) \xrightarrow{\pi} C_{Re \geq 0}$

Take  $J = J_F + J_G$  cyl,  $u: \Sigma \rightarrow X$   $J$  hol

Lemma: If  $\Sigma$  connected, and  $u$  map  
 $\partial\Sigma$  or  $Nbd$  (punctures) in  $X \setminus F_X \cap_{Re \leq \epsilon}$   
then  $u$  doesn't intersect  $F_X \cap_{Re \leq \epsilon}$

Pf Notice  $u^{-1}(F_X \cap_{Re \leq \epsilon})$  compact

$\hookrightarrow Y := \text{Image} (\pi \circ u|_{u^{-1}(F_X \cap_{Re \leq \epsilon})})$

$Y$  bounded and closed in  $F_X \cap_{Re \leq \epsilon}$

Now:  $\pi \circ u$  cannot be constant

bec.  $\partial\Sigma / Nbd$  (punct)  $\not\subset F_X \cap_{Re \leq \epsilon}$

Then open map from  $Y$  open

$$\boxed{\Rightarrow \partial = \emptyset}$$

By

## Cutoff Reeb dynamics

Let  $(Y, \xi)$  std. bdry at  $\infty$  of L-sector  $X$

In part  $\partial Y$  convex hypersurface in  $Y$

Let  $\alpha$  std.  $\xi = \ker \alpha$  and  $\rho: Y \rightarrow \mathbb{R} \leq 0$  defining  $\partial Y$   
 (i.e.  $\rho$  has  $\neq 0$  zeros on  $\partial Y$ ,  $\rho < 0$  on  $\overset{\circ}{Y}$ )

Def Cutoff Reeb vector field as a contact v.f.

on  $Y$  with contact Hamiltonian (w.r.t.  $\alpha$ )

$\ell^2 \cdot H$ , for some  $H: Y \rightarrow \mathbb{R} > 0$

[Recall: std v.f.  $V_F$  is given by  
 $d(V_F) = F$ ,  $i_{V_F} d\alpha|_{\xi} = -dF|_{\xi}$ ]

Lemma | Let  $R$  cutoff R.v.f. Then  $\exists K \subset Y \setminus \partial Y$  cpt  
 s.t.  $R$  intersects every closed orbit of  $R$  (besides those)  
 [Idea: find  $G: \text{Nbd}(\partial Y) \rightarrow \mathbb{R}$  s.t.  $dG(R) > 0$ ]

(will need more control for some specific cutoff R.v.f.)

Lemma If  $M: (0, \varepsilon) \rightarrow \mathbb{R}_{\geq 0}$  is  $\equiv 0$  at  $t=0$ ,  
 normal coord w.r.t.  $\partial Y$  and  $M''(0) > 0$ ,  $M' > 0$  for  $t > 0$ .  
 Then  $V_M$  has this dynamics near  $\partial Y$ !

