

A degenerate Morse-to-CZ index relation and global surfaces of section for non-convex billiards

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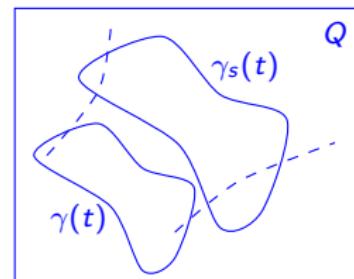
0. What is the Morse index of a periodic orbit and why is it finite?

(Q, g) any Riemannian mfd. with a potential $U \in C^\infty(Q)$

$\gamma \in C^\infty(S^1, Q)$ loop of fixed period \rightsquigarrow action $S[\gamma] = \int_{S^1} \frac{1}{2} \|\dot{\gamma}\|^2 - U(\gamma)$

$V \in \Gamma(\gamma^* TQ)$ vector field along $\gamma \rightsquigarrow \gamma(s, t) = \exp_{\gamma(t)} s \cdot V(t)$ family of loops with $\partial_s \gamma(0, t) = V(t)$

Study $S[\gamma_s]$ as a function of s !



First variation

$$\partial_s S[\gamma_s]|_{s=0} = - \int g(\mathcal{F}(\gamma), V) \stackrel{!}{=} 0 \text{ for all } V \iff \gamma \text{ satisfies e.o.m. } \mathcal{F}(\gamma) = \frac{D}{dt} \dot{\gamma} + \nabla U = 0$$

Second variation

$$\partial_s^2 S[\gamma_s]|_{s=0} = - \int g\left(\mathcal{F}(\gamma), \underbrace{\frac{D}{\partial s} \partial_s \gamma}_{=0 \text{ on-shell}}\right) + g\left(\frac{D}{\partial s} \mathcal{F}(\gamma), V\right) = I(V, V)$$

where $\mathcal{L} \subset \Gamma(\gamma^* TQ)$, $\mathcal{L}V = \frac{D}{\partial s} \mathcal{F}(\gamma_s)|_{s=0} = \left(\frac{D}{\partial t}\right)^2 V + R(V, \dot{\gamma})\dot{\gamma} + \nabla_V \nabla U$ linearized e.o.m.

$$I(V, W) = - \int g(\mathcal{L}V, W) = \int g\left(\frac{DV}{\partial t}, \frac{DW}{\partial t}\right) - g(R(V, \dot{\gamma})\dot{\gamma} + \nabla_V \nabla U, W) \text{ index form}$$

Def. $Morse(\gamma) = \sup \dim E, \quad E \subset \Gamma(\gamma^* TQ) \text{ such that } I|_E \times E \text{ negative-definite}$

$\kappa \gg 0$ large enough \rightsquigarrow

$$r_1 g(V, V) \leq \kappa g(V, V) - g(R(V, \dot{\gamma})\dot{\gamma} + \nabla_V \nabla U, V) \leq r_2 g(V, V) \quad \text{for some } r_1, r_2 > 0$$

$L^2(S^1, \gamma^* TQ)$ with inner product $G(V, W) = \kappa \int g(V, W)$

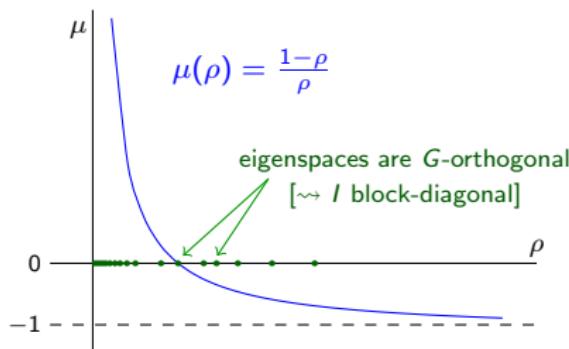
$W^{1,2}(S^1, \gamma^* TQ)$ with inner product $(I + G)(V, W) = \int g\left(\frac{\partial V}{\partial t}, \frac{\partial W}{\partial t}\right) + \dots$

Riesz representation thm $\implies \exists! K \in \mathcal{L}(L^2, W^{1,2}) : (I + G)(KV, \bullet) = G(V, \bullet)|_{W^{1,2}}$

$$\begin{array}{ccc} L^2 & \xrightarrow{K_0} & L^2 \\ \uparrow K & \searrow & \uparrow \text{compact} \\ W^{1,2} & \xrightarrow{K_1} & W^{1,2} \end{array}$$

& $(I+G)(KV, KW) = G(V, K_0 W)$
 $G(V, W) = (I+G)(V, K_1 W) \implies K_0, K_1 \text{ compact, self-adjoint, positive symmetric, pos.def.}$

$$KV = \rho V \iff V \in W^{1,2} \text{ and } I(V, \bullet) = \mu(\rho) G(V, \bullet)|_{W^{1,2}} \iff V \in C^\infty \text{ and } [\mathcal{L} + \mu\kappa]V = 0$$



Hilbert-Schmidt for K_1 :

$$W^{1,2} = \overline{\bigoplus_{\rho \leq 1} \ker[K - \rho]} \oplus \overline{\bigoplus_{\rho > 1} \ker[K - \rho]}$$

$E_{\geq 0}$ / non-negative E_- / negative def.

$$I|_E < 0 \implies E \cap E_{\geq 0} = 0 \implies \text{pr}_- : E \hookrightarrow E_-$$

Conclusion: $Morse(\gamma) = \dim E_- = \sum_{\mu < 0} \ker[\mathcal{L} + \mu\kappa] < \infty$

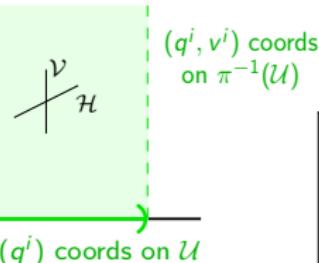
1. From Lagrange to Hamilton

$$M = TQ \cong T^* Q$$

phase space

π

position space Q



$$f_i(q, v) = \frac{\partial}{\partial v^i}$$

$$e_i(q, v) = \frac{\partial}{\partial q^i} - \Gamma_{ij}^k(q)v^j \frac{\partial}{\partial v^k}$$

vector fields
on $\pi^{-1}(U)$

Under $(\tilde{q}(q), \frac{\partial \tilde{q}^i}{\partial q^k} v^k)$ coord change on M :

$$e_i = \frac{\partial \tilde{q}^k}{\partial q^i} \tilde{e}_k \text{ and } f_i = \frac{\partial \tilde{q}^k}{\partial q^i} \tilde{f}_k \text{ transform block-diagonally}$$

↔ Globally $TM = \mathcal{H} \oplus \mathcal{V} = \text{span}\{e_i\} \oplus \text{span}\{f_i\}$

Horizontal and vertical lift

$$\alpha^i(q, v) \frac{\partial}{\partial q^i} \in \Gamma(\pi^* TQ) \rightsquigarrow \alpha^{\mathcal{H}} = \alpha^i(q, v) e_i(q, v), \quad \alpha^{\mathcal{V}} = \alpha^i(q, v) f_i(q, v)$$

Examples: $v^{\mathcal{H}} = v^i \frac{\partial}{\partial q^i} - \Gamma_{ij}^k v^i v^j \frac{\partial}{\partial v^k}$ geodesic vf. $X_H = v^{\mathcal{H}} - [\nabla U]^{\mathcal{V}}$ Hamiltonian vf.

$$\gamma(t) \in Q \text{ solution of } \frac{D}{dt} \dot{\gamma} + \nabla U = 0 \quad \overset{1:1}{\longleftrightarrow} \quad x(t) = (\gamma(t), \dot{\gamma}(t)) \in TQ \text{ flow line of } X_H$$

Linearised flow equation

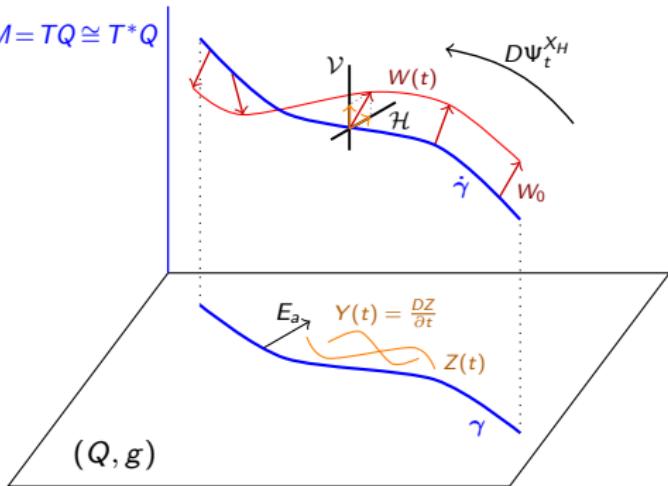
Study $\mathcal{F}(x) = \partial_t x - X_H(x) \in \Gamma(x^* TM)$ on a family $x(s, t)$ with $\partial_s x(0, t) = W(t)$:

$$\frac{\bar{D}}{\partial s} \mathcal{F}(x_s) \Big|_{s=0} = \frac{\bar{D}W}{\partial t} - \bar{\nabla}_W X_H = \underbrace{\left[\partial_t W^\mu - \frac{\partial X_H^\mu}{\partial x^\nu} W^\nu \right]}_{\mathcal{D}W} \frac{\partial}{\partial x^\mu} + \underbrace{\bar{\Gamma}_{\nu\kappa}^\mu W^\kappa [\dot{x}^\nu - X_H^\nu]}_{=0 \text{ on-shell}} \frac{\partial}{\partial x^\mu}$$

↔ covariant derivative $\mathcal{D} \cap \Gamma(x^* TM)$

$$\mathcal{D}W = [X_H, W] \text{ on ambient vf. and } \mathcal{D}(D\Psi_t^{X_H}(x_0)W_0) = 0 \text{ for any initial } W_0 \in T_{x_0} M$$

$$M = TQ \cong T^*Q$$



$\omega = dp_i \wedge dq^i$ symplectic form on T^*Q

For any $Y, Z \in T_q Q$:

$$\omega(Y^H, Z^H) = \omega(Y^V, Z^V) = 0$$

$$\omega(Y^V, Z^H) = g(Y, Z)$$

$\{E_a\}$ ONB of $T_q Q \rightsquigarrow$
 $\{F_a\} = \{E_a^V, E_a^H\}$ sympl. basis on $T_{(q,v)} M$

$$D\Psi_t^{X_H}(x_0) F_a(0) = F_b(t) \Phi_{ba}(t)$$

with $\Phi(t) \in Sp(\mathbb{R}^{2n})$

Relation between $D\Psi_t^{X_H}$ and the linearised Euler-Lagrange equation

Given any $W_0 \in T_{x_0} M$ write $W(t) = D\Psi_t^{X_H}(x_0) W_0 = E_a(t)^V Y^a(t) + E_a(t)^H Z^a(t) = Y^V + Z^H$

$$0 = \mathcal{D}[D\Psi_t^{X_H}(x_0) W_0] = \mathcal{D}(Y^V + Z^H) = \left[\frac{DY}{dt} + R(Z, \dot{\gamma})\dot{\gamma} + \nabla_Z \nabla U \right]^V + \left[\frac{DZ}{dt} - Y \right]^H$$

$[e_i, e_j] = -R_{ijr}^k v^r f_k$ etc.

Conclusion: $Z(t) = \text{pr}_H[D\Psi_t^{X_H}(x_0) W_0]$ solves $\left[\frac{D}{dt} \right]^2 Z + R(Z, \dot{\gamma})\dot{\gamma} + \nabla_Z \nabla U = 0$
 and one recovers $\frac{DZ}{dt} = \text{pr}_V[D\Psi_t^{X_H}(x_0) W_0]$

- Write $\mathcal{D} \circ \Gamma(x^* TM)$ in $\begin{bmatrix} \mathcal{V} \\ \mathcal{H} \end{bmatrix}$ -form
- Introduce a perturbation μ
- Use E_a -trivialisation on both components (\rightsquigarrow corresponds to $\{E_a^\mathcal{V}, E_a^\mathcal{H}\}$ -trivialisation on TM)

$$\begin{aligned}\mathcal{D}_\mu(Y^\mathcal{V} + Z^\mathcal{H}) &= \begin{bmatrix} \frac{DY}{\partial t} \\ \frac{DZ}{\partial t} \end{bmatrix} + \begin{bmatrix} -\text{id} & R(\bullet, \dot{\gamma})\dot{\gamma} + \nabla_\bullet \nabla U + \mu \kappa \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} \\ &= \begin{bmatrix} \dot{Y}_a \\ \dot{Z}_a \end{bmatrix} + \begin{bmatrix} \Omega_{ab} & g(E_a, R(E_b, \dot{\gamma})\dot{\gamma} + \nabla_{E_b} \nabla U + \mu \kappa E_b) \\ -\delta_{ab} & \Omega_{ab} \end{bmatrix} \begin{bmatrix} Y_b \\ Z_b \end{bmatrix} \\ &= \begin{bmatrix} \dot{Y}_a \\ \dot{Z}_a \end{bmatrix} - \begin{bmatrix} -\text{id} \\ \text{id} \end{bmatrix} \begin{bmatrix} \delta_{ab} & -\Omega_{ab} \\ \Omega_{ab} & g(E_a, R(E_b, \dot{\gamma})\dot{\gamma} + \nabla_{E_b} \nabla U + \mu \kappa E_b) \end{bmatrix} \begin{bmatrix} Y_a \\ Z_a \end{bmatrix} = \dot{v} - JS(\mu, t)v\end{aligned}$$

where $\frac{DE_b}{\partial t} = E_a \Omega_{ab} \implies \Omega_{ab} = g(E_a, \frac{DE_b}{\partial t}) = -\Omega_{ba}$

fundamental solution $\Phi(\mu, t) \in Sp(\mathbb{R}^{2n}, \omega_{std})$
 $\dot{\Phi}_\mu - JS(\mu, t)\Phi_\mu = 0, \quad \Phi(\mu, 0) = \text{id}$

We will exploit the following properties of $\Phi(\mu, t)$:

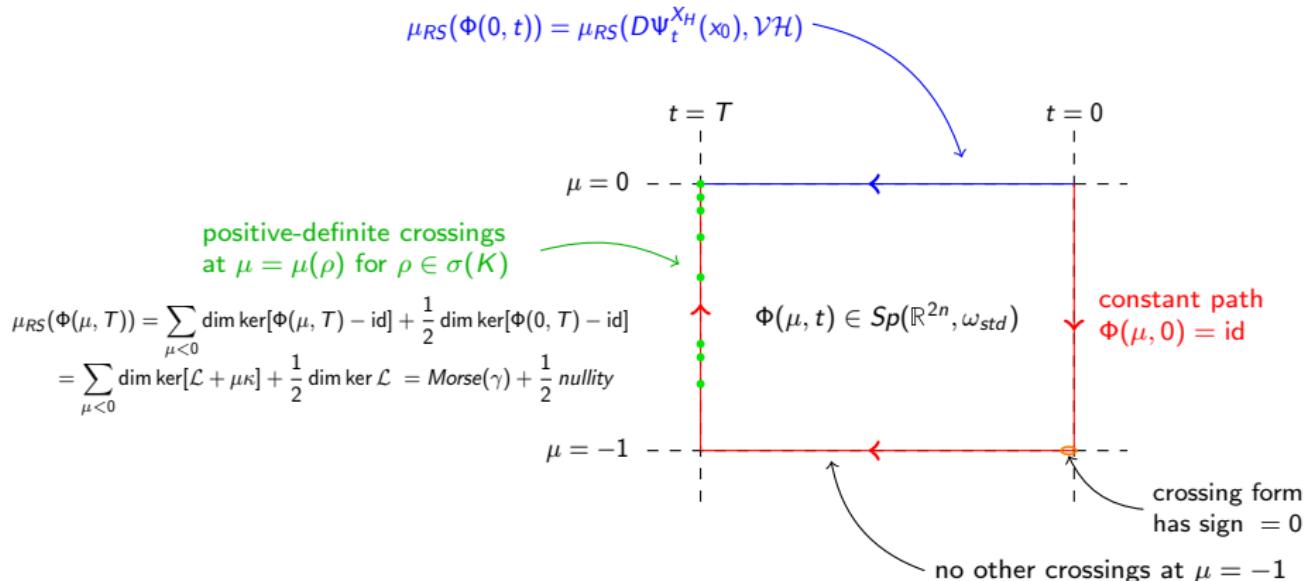
1) $\Phi_\mu(t+T) = \Phi_\mu(t)\Phi_\mu(T)$, so $v(t) = \Phi(\mu, t)v(0)$ T-periodic $\iff v(0) \in \ker[\Phi(\mu, T) - \text{id}]$

At every $\mu \in \mathbb{R}$: $\ker[\Phi(\mu, T) - \text{id}] \xrightarrow{\sim} \ker[\mathcal{L} + \mu \kappa]$, $v \mapsto \text{pr}_{\mathcal{H}}[\Phi(\mu, t)v]$

linear iso with inverse $(\frac{DZ}{\partial t}(0), Z(0)) \longleftrightarrow Z$

2) $\Phi(t) = \Phi(0, t) \in Sp(\mathbb{R}^{2n})$ trivialises $D\Psi_t^{X_H}(x_0)$ w.r.t. $\{E_a^\mathcal{V}, E_a^\mathcal{H}\}$

Duistermaat's square (using the Robbin-Salamon index)

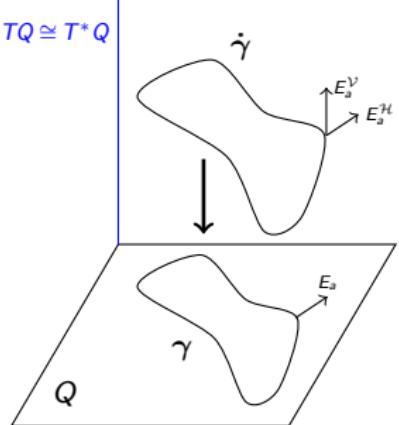


By homotopy invariance and catenation properties of the RS index:

$$\boxed{\mu_{RS}(D\Psi_t^{X_H}(x_0), \mathcal{V}\mathcal{H}) = \text{Morse}(\gamma) + \frac{1}{2} \text{nullity}}$$

Change of trivialisation

$$M = TQ \cong T^*Q$$

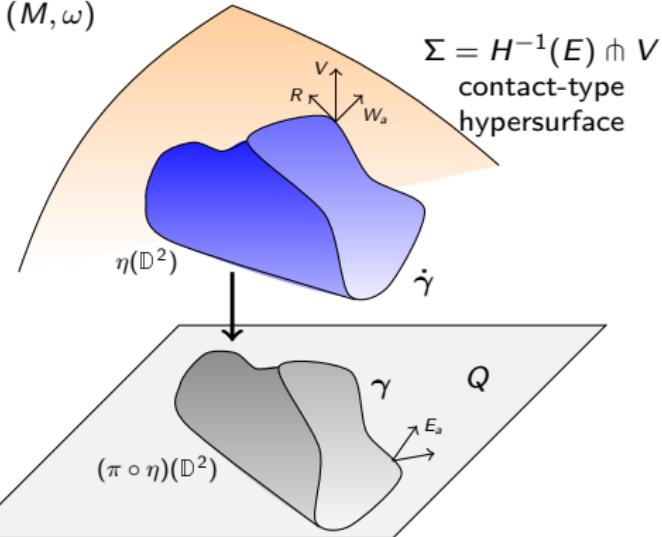


$E_a(t + T) = E_a(t)$
periodic orthonormal frame
along γ

$\rightsquigarrow \{E_a^V, E_a^H\}$ periodic symplectic frame along $\dot{\gamma}$

For any frame of this type:

$$\mu_{RS}(D\Psi_t^{X_H}(x_0), \{E_a^V, E_a^H\}) = \text{Morse}(\gamma) + \frac{1}{2} \text{nullity}$$



V Liouville vf. transverse to $\Sigma \rightsquigarrow \omega(V, \cdot) = \lambda$ restricts to a contact form
 R Reeb vf. $\lambda(R) = 1, \omega(R, \cdot)|_{T\Sigma} = 0$

1. Assume sympl. triv. W_a of $\xi = \ker \lambda|_{T\Sigma}$
extends along $\eta : \mathbb{D}^2 \rightarrow \Sigma$

$\rightsquigarrow \{F_a\} = \{V, R, W_a\}$ sympl. triv. of $\eta^* TM$

2. Choose orthonormal triv. E_a of TQ along $\pi \circ \eta$

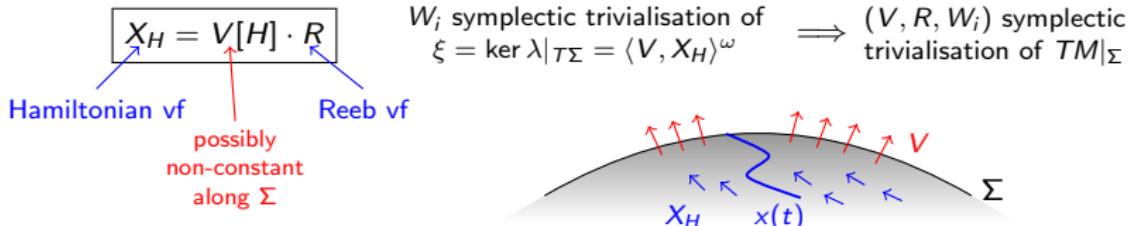
$\rightsquigarrow \{F'_a\} = \{E_a^V, E_a^H\}$ another sympl. triv. of $\eta^* TM$

3. $F'_a = F_b \varphi_{ba}$ contractible loop in $Sp(\mathbb{R}^{2n}, \omega_{std})$

$$\mu(D\Psi_t^{X_H}(x_0), F) - \mu(D\Psi_t^{X_H}(x_0), F') = 2 \mu_{Mas}(\varphi) = 0$$

2. From Hamilton to Reeb

(M, ω) symplectic mfd. $\Sigma = H^{-1}(E) \pitchfork V$, $\omega = d(\frac{\lambda}{\iota_V \omega})$ contact-type hypersurface



	$V_{x(0)}$	$R_{x(0)}$	$W_i(0)$	
linearized Hamilton flow $\Phi(t) =$	$V_{x(t)}$	$a(t) \quad 0$	$0 \cdots 0$	$\rho(t, x) = \int_0^t dt' V[H] \circ \Psi_{t'}^{X_H}(x)$
	$R_{x(t)}$	$b(t) \quad \frac{1}{a(t)}$	$e_i(t)$	$\Psi^{X_H}(t, x) = \Psi^R(\rho(t, x), x)$
	$W_j(t)$	$c_j(t) \quad 0 \quad \vdots \quad 0$	$\Theta(t)_{ji}$ linearized Reeb flow	$a(t) = \frac{V[H]_{x(0)}}{V[H]_{x(t)}}$ satisfies $a(0) = a(T) = 1$ $b(t) = V[\rho_t - t \cdot H]_{x(0)}$ $e_i(t) = d\rho_t _{x(0)} W_i(0)$ $c_j(t) \omega_{jk} \Theta(t)_{ki} + a(t) \cdot e_i(t) = 0$

Claim: Depending on $N = \dim \ker[\Phi(T) - \text{id}] - \dim \ker[\Theta(T) - \text{id}]$

$$k = \mu_{RS}(\Phi) - \mu_{RS}(\Theta) = \begin{cases} 0 & N = 0, 2 \\ \pm \frac{1}{2} & N = 1 \end{cases}$$

Graphical proof of

$$N = 0, 2 \implies k = 0$$

$$\Phi_{\delta,r}(t) = \left[\begin{array}{cc|c} \delta \cdot a(t) & 0 & 0 \cdots 0 \\ \delta \cdot r^2 \cdot b(t) & \frac{1}{\delta \cdot a(t)} & r \cdot e(t) \\ \hline \delta \cdot r \cdot c(t) & 0 & \Theta(t) \\ \vdots & 0 & \\ \end{array} \right] \quad \text{family of symplectic matrices}$$

Key observation:

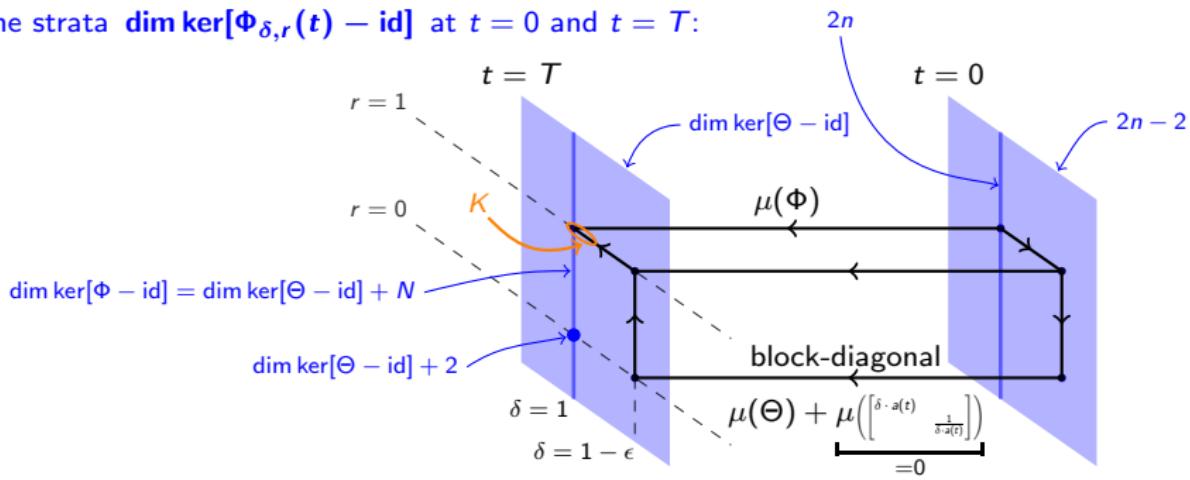
$$\ker[\Phi_{\delta,r}(T) - \text{id}] = \ker \left[\begin{array}{c|c|c} \delta - 1 & & \\ \hline \delta r^2 \cdot b & \frac{1}{\delta} - 1 & r \cdot e \\ \hline \delta r \cdot c & & \Theta - \text{id} \end{array} \right] = \begin{cases} [0, \frac{\delta r}{\delta - 1} \langle e, \bullet \rangle, \text{id}] \ker[\Theta(T) - \text{id}] & \delta \neq 1 \\ \text{diag}[1, 1, r] \ker[\Phi(T) - \text{id}] & \delta = 1, r \neq 0 \\ \mathbb{R}^2 \oplus \ker[\Theta(T) - \text{id}] & \delta = 1, r = 0 \end{cases}$$

Graphical proof of

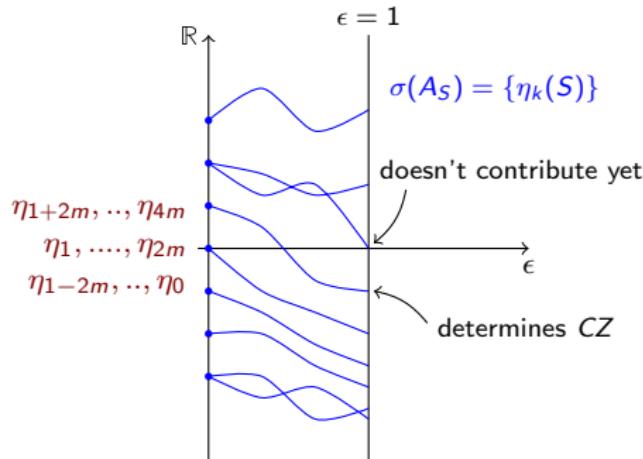
$$N = 0, 2 \implies k = 0$$

$$\Phi_{\delta,r}(t) = \left[\begin{array}{cc|c} \delta \cdot a(t) & 0 & 0 \cdots 0 \\ \delta \cdot r^2 \cdot b(t) & \frac{1}{\delta \cdot a(t)} & r \cdot e(t) \\ \hline \delta \cdot r \cdot c(t) & 0 & \Theta(t) \\ \vdots & \vdots & \\ 0 & 0 & \end{array} \right] \quad \text{family of symplectic matrices}$$

Study the strata $\dim \ker[\Phi_{\delta,r}(t) - \text{id}]$ at $t = 0$ and $t = T$:



3. From Robbin-Salamon to Conley-Zehnder



Pick any continuous path S_ϵ interpolating between $S_0 = 0$ and $S_1 = S$ [e.g. $S_\epsilon = \epsilon \cdot S$]

Study family of continuous sections

$\eta_k : C^\infty(S^1, Sym(\mathbb{R}^{2m})) \rightarrow \mathbb{R}$ such that

- $\eta_k \leq \eta_{k+1}$
- $\dim \ker[A_S - \eta] = \#\{k \mid \eta_k(S) = \eta\}$
- $\eta_k(0) = 0$ for $k \in \{1, \dots, 2m\}$

Definition of the spectral CZ index

To $\Theta : \mathbb{R} \rightarrow Sp(\mathbb{R}^{2m})$
w/ $\Theta(0) = \text{id}$ associate

$$S = J_0 \dot{\Theta} \Theta^{-1} : \mathbb{R} \rightarrow Sym(\mathbb{R}^{2m})$$

$\Theta(t+T) = \Theta(t)\Theta(T) \iff S(t+T) = S(t)$
monodromy relation periodicity

To $S \in C^\infty(S^1, Sym(\mathbb{R}^{2m}))$ associate
 $A_S = J_0 \partial_t - S : W^{1,2}(S^1, \mathbb{R}^{2m}) \rightarrow L^2(S^1, \mathbb{R}^{2m})$

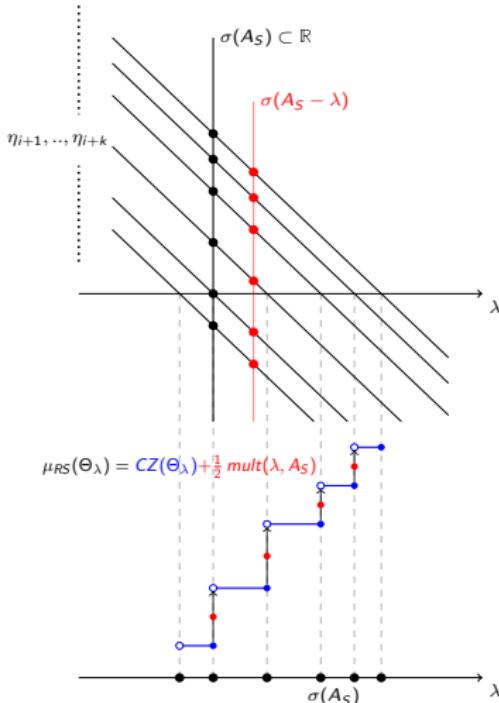
$$CZ(\Theta) = \max \{ k \in \mathbb{Z} \mid \eta_k(S) < 0 \} - m$$

Relation between CZ and μ_{RS}

For non-degenerate paths:

Prop. ([HWZ] Prop.3.3 [FvK] Thm 11.2.1)

$$\ker[\Theta(T) - \text{id}] = 0 \implies \mu_{RS}(\Theta) = \text{CZ}(\Theta)$$



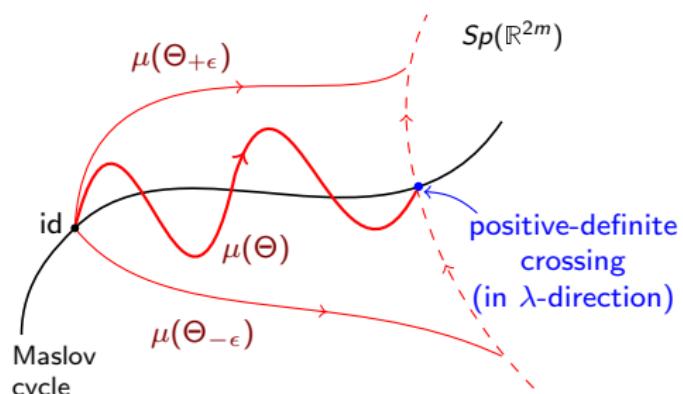
For degenerate paths:

$$\mu_{RS}(\Theta) = \text{CZ}(\Theta) + \frac{1}{2} \dim \ker[\Theta(T) - \text{id}]$$

Proof: Study perturbations $A_S \rightarrow A_S - \lambda$

$$[J_0 \partial_t - (S + \lambda)] \Theta_\lambda(t) = 0 \quad \Theta_\lambda(0) = \text{id}$$

fundamental solution



$$\mu_{RS}(\Theta) = \mu_{RS}(\Theta_{-\epsilon}) + \frac{1}{2} \dim \ker[\Theta(T) - \text{id}]$$

$$\mu_{RS}(\Theta_{-\epsilon}) = \text{CZ}(\Theta_{-\epsilon}) = \text{CZ}(\Theta)$$

□

SUMMARY: Degenerate Morse-to-CZ index relation

nullity difference $N = \dim \ker[\Phi(T) - \text{id}] - \dim \ker[\Theta(T) - \text{id}] \in \{0, 1, 2\}$

(full) nullity transverse nullity

$$\text{Morse} + \frac{\text{nullity}}{2} = \mu_{RS}(\Phi)$$

[Duistermaat '76 + ϵ]

ϵ = Robbin-Salamon index ('92)
+ differential geometry [incl. change
of symplectic trivialisation]

$$\mu_{RS}(\Phi) = \mu_{RS}(\Theta) + k \quad \text{← correction term}$$

$$\mu_{RS}(\Theta) = CZ + \frac{\text{transverse nullity}}{2}$$

[perturb asymptotic operator
to make it non-degenerate
+ study the resulting crossing form]

$$\implies \boxed{\text{Morse} + \frac{N}{2} = CZ + k}$$

Moreover

$$\begin{aligned} N = 0, 2 &\implies k = 0 \\ N = 1 &\implies k = \pm \frac{1}{2} \end{aligned}$$

and thus always

$$CZ - \text{Morse} \in \{0, 1\}$$

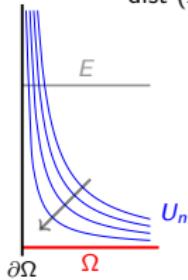
4. Global surfaces of section for non-convex billiards

Given a smooth potential U , point particles move according to Newton's equation

$$\ddot{\gamma} + \nabla U = 0 \quad [\Rightarrow \text{energy } E = \frac{1}{2}\dot{\gamma}^2 + U(\gamma) \text{ is conserved}]$$

Two approximation schemes modelling dynamics on a billiard table $\Omega \subset \mathbb{R}^m$

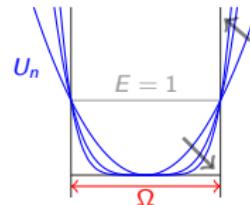
1) [AM] $U_n \sim \frac{\epsilon_n}{\text{dist}(x, \partial\Omega)^2}$, $\epsilon_n \rightarrow 0$



2) For starshaped Ω :

$$\Omega = \tilde{U}^{-1}(<1) \quad \partial\Omega = \tilde{U}^{-1}(1) \text{ with } \tilde{U}(rx) = r^2 \tilde{U}(x)$$

$$U_n := \tilde{U}^n \text{ preserves homogeneity}$$



COMMON RECIPE

INPUT:

γ_n periodic orbit for U_n
with period $\tau_- < \tau < \tau_+$
and energy $0 < E < E_+$

UNIVERSAL

energy bound
 $\Rightarrow \gamma, \dot{\gamma} \in \mathbb{R}^m$ uniformly bounded

**SPECIFIC to
approximation scheme**

energy bound + equation of motion
 $\Rightarrow L^1$ -bound on $\nabla U|_\gamma \Rightarrow L^1$ -bound on $\ddot{\gamma}$

$$q(s) = \gamma(\tau s)$$

\uparrow \uparrow
1-periodic τ -periodic

$$q_n \in W^{2,1}(S^1, \mathbb{R}^m) \xrightarrow[\text{compact}]{\text{bounded}} W^{1,2} \hookrightarrow C^0$$

$$\Rightarrow \exists \text{ subsequence } q_n \xrightarrow{W^{1,2}} q_\infty$$

Both schemes are valid approximations of the billiard dynamics in the sense that:

i) $\nabla U_n|_{q_n}$ is bounded in L^1

up to subsequence
 \implies

limit $q_n \rightarrow q_\infty \in W^{1,2}(S^1, \mathbb{R}^m)$ exists

and by Banach-Alaoglu + Riesz-Markov we have
a positive Borel measure μ on the domain such that

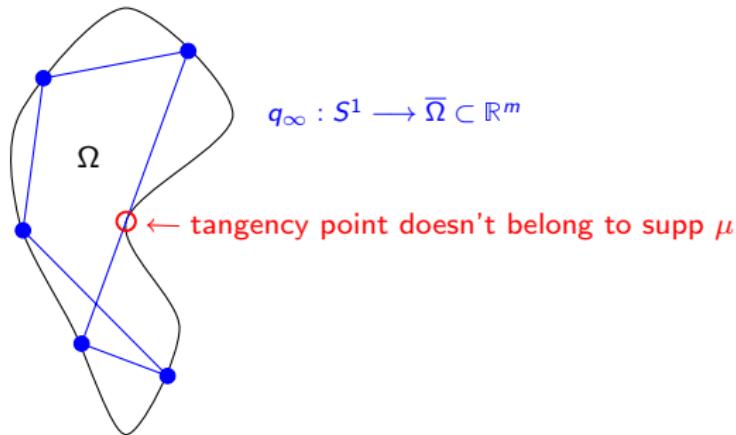
$$\int_{S^1} \|\nabla U_n|_{q_n}\| \cdot f \rightarrow \int_{S^1} d\mu f \quad \forall f \in C^0(S^1, \mathbb{R})$$

ii) On $S^1 \setminus \text{supp } \mu$: q_∞ consists of straight lines

iii) At isolated points of $\text{supp } \mu$: q_∞ changes direction according to the law of reflection

iv) $\#\text{supp } \mu \leq \liminf \text{Morse}(\gamma_n)$

\implies If R.H.S. is finite: $\text{supp } \mu = \text{finite set consisting of } \underline{\text{'bounce points'}}$



$$q_\infty : S^1 \longrightarrow \bar{\Omega} \subset \mathbb{R}^m$$

SUMMARY

closed subsets

$$\text{supp } \mu \subset q_\infty^{-1}(\partial\Omega) \subset S^1$$

\uparrow \uparrow
 q_∞ changes q_∞ touches
direction the boundary

ASIDE Simplifications arising for approximation scheme (2):

Given a homogeneous potential $U(rq) = r^n U(q)$ use the Liouville vector field

$$V = \frac{n}{n+2} \cdot p_i \frac{\partial}{\partial p_i} + \frac{2}{n+2} \cdot q_i \frac{\partial}{\partial q_i}$$

Applied to $H = \frac{1}{2}|p|^2 + U(q)$ this gives

$$V[H] = \frac{2n}{n+2} \cdot H$$

$$\rho_t = \int_0^t dt' V[H] \circ \Psi_{t'}^{X_H} = \frac{2n}{n+2} \cdot t H$$

- $d\rho_t|_\Sigma = 0 \implies e(t), c(t) = 0$ i.e. $\Phi(t)$ is block-diagonal]

$$\begin{aligned} \bullet b(t) &= V[\rho_t - t \cdot H] = \frac{2n}{n+2} \underbrace{\left[\frac{2n}{n+2} - 1 \right]}_{\begin{aligned} &= 0 \quad n=2 \\ &> 0 \quad n>2 \end{aligned}} \cdot t H \end{aligned}$$

Since we are interested in the limit of large n :

$$k = \mu_{RS}(\Phi) - \mu_{RS}(\Theta) = \mu \begin{bmatrix} 1 \\ b(t) & 1 \end{bmatrix} = +\frac{1}{2}$$

$$N = 1$$

$$\xrightarrow{Morse+N/2=CZ+k} \boxed{\text{Morse} = \text{CZ}} \text{ for all orbits}$$

Alternative proof of a classical result by Benci-Giannoni

(1) $U_n \sim \frac{\epsilon_n}{\text{dist}(x, \partial\Omega)^2}$ with $\epsilon \rightarrow 0$ or (2) $U_n = \tilde{U}^n$ with $\tilde{U}(rq) = r^2 \tilde{U}(q)$ homogeneous

$H_n = \frac{1}{2}|p|^2 + U_n(q)$, $\Sigma_n = H_n^{-1}(1)$ contact-type hypersurface (w.r.t. n -dependent Liouville vf.)

Prop. Assume Ω has no 2-bounce orbits.

Then for every $T > 0$ there exists $N(T)$ such that the Reeb flow on $\Sigma_{n>N}$ is **dynamically convex up to period T** (i.e. every closed orbit of period $< T$ has CZ index ≥ 3)

Proof: Suppose we can find a subsequence γ_n with bounded period and $CZ(\gamma_n) \leq 2$.

Then we can extract a limit orbit $q_\infty : S^1 \longrightarrow \bar{\Omega}$ with $\#\text{bounce pts.} \leq \liminf \text{Morse}(\gamma_n) \leq 2$ ↗

Def. Given $\Sigma \cong S^3$ with a tight contact form λ

consider $T_{crit} \sim \inf_{\Psi} \max_{x \in S^3} f$ where $\Psi : S^3 \xrightarrow{\sim} \Sigma$, $\Psi^* \lambda = f \lambda_{Std} + \text{exact}$

Using [HWZ]: (Σ, λ) dynamically convex up to period T_{crit}

⇒ Reeb flow admits a disk-like GSS whose binding has $CZ = 3$ and period $< T_{crit}$

Corollary (cf. [BG], [AM]) Assume $T_{crit}(n)$ stays bounded as $n \rightarrow \infty$.

Then the billiard table Ω admits a periodic orbit with **at most 3 reflections**.

Proof: Write $T_\infty = \sup T_{crit}(n) < \infty$ and suppose Ω doesn't have 2-bounce orbits.

Then for sufficiently large n :

(Σ_n, λ_n) dyn. conv. up to period $T_\infty \implies$ Reeb flow admits GSS w/ binding γ_n

Since γ_n has bounded period and $CZ(\gamma_n) = 3$:

repeat argument from above to obtain billiard orbit with 3 reflections □

Why does $T_{crit}(n)$ stay bounded in the limit $n \rightarrow \infty$?

$$T_{crit}(\Sigma, \alpha) := \inf_{\Psi} \max_{x \in S^3} f \quad \text{where } \Psi : S^3 \xrightarrow{\sim} \Sigma \text{ diffeo w/ } \Psi^* \alpha = f \lambda_{std}$$

Useful properties:

a) $T_{crit}(\Sigma, \alpha + dh) = T_{crit}(\Sigma, \alpha)$ [by Gray stability]

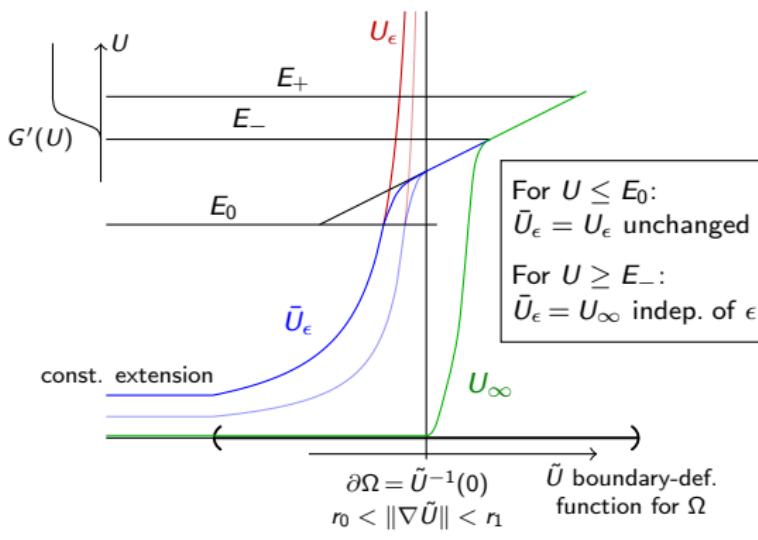
$\rightsquigarrow T_{crit}$ is the same for any choice of transverse Liouville vf.

b) $e^{\min(h)} T_{crit}(\Sigma, \alpha) \leq T_{crit}(\Sigma, e^h \alpha) \leq e^{\max(h)} T_{crit}(\Sigma, \alpha)$

$\rightsquigarrow T_{crit}$ is monotonic under Liouville flow-out

Claim: $T_{crit}(n)$ stays bounded, also in approximation scheme (1)

Proof



1. At fixed ϵ :

$$V_\epsilon = p_i \frac{\partial}{\partial p_i} + X_{k_\epsilon}$$

$$k = \delta(\epsilon) \langle p, \nabla \bar{U}_\epsilon \rangle \quad \delta(\epsilon) > 0 \text{ suff. small}$$

⇒ transverse Liouville flow-out
connecting $\bar{H}_\epsilon^{-1}(E_0)$ and $\bar{H}_\epsilon^{-1}(E_+)$

2. Change transverse Liouville vf.

along $\Sigma_\epsilon^+ = \bar{H}_\epsilon^{-1}(E_+)$:

$$V_\infty = p_i \frac{\partial}{\partial p_i} + X_k$$

$$k = \delta \langle p, \nabla G(U_\infty) \rangle \text{ indep. of } \epsilon$$

⇒ $\lambda_\infty = p_i dq^i$ for $q \in U_\infty^{-1}(\leq E_-)$

3. $\varphi : \Sigma_\epsilon^+ \xrightarrow{\sim} \Sigma_\infty \quad (q, p) \mapsto (q, f(q)p)$

where $f(q) = \sqrt{\frac{E_+ - U_\infty}{E_+ - \bar{U}_\epsilon}} \geq 1$

and $f = 1$ for $q \notin U_\infty^{-1}(\leq E_-)$

⇒ $\varphi^* \lambda_\infty = f \lambda_\infty$

Conclusion: $T_{crit}(\Sigma_\epsilon) \stackrel{(1)}{\leq} T_{crit}(\Sigma_\epsilon^+, \lambda_\epsilon) \stackrel{(2)}{=} T_{crit}(\Sigma_\epsilon^+, \lambda_\infty) \stackrel{(3)}{\leq} T_{crit}(\Sigma_\infty)$ uniformly bounded □